

1. Solve the equation

$$7\operatorname{sech} x - \tanh x = 5$$

Give your answers in the form $\ln a$ where a is a rational number.

(5)

$$7\operatorname{sech} x - \tanh x = 5$$

$$\therefore \frac{7}{\cosh x} - \frac{\sinh x}{\cosh x} = 5$$

$$\therefore 7 - \sinh x = 5 \cosh x$$

$$\therefore 7 = 5 \cosh x + \sinh x$$

$$\therefore 7 = \frac{5e^x + 5e^{-x}}{2} + \frac{e^x - e^{-x}}{2}$$

$$\therefore 14 = 6e^x + 4e^{-x}$$

$$\therefore 14e^x = 6e^{2x} + 4$$

$$\therefore 6e^{2x} - 14e^x + 4 = 0 \Rightarrow 3e^{2x} - 7e^x + 2 = 0$$

$$(e^x - 2)(3e^x - 1) = 0$$

$$e^x = 2 \Rightarrow x = \ln 2$$

$$e^x = \frac{1}{3} \Rightarrow x = \ln \frac{1}{3}$$



2.

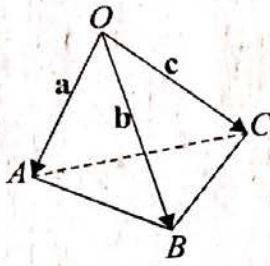


Figure 1

The points A , B and C have position vectors \mathbf{a} , \mathbf{b} and \mathbf{c} respectively, relative to a fixed origin O , as shown in Figure 1.

It is given that

$$\mathbf{a} = \mathbf{i} + \mathbf{j}, \quad \mathbf{b} = 3\mathbf{i} - \mathbf{j} + \mathbf{k} \quad \text{and} \quad \mathbf{c} = 2\mathbf{i} + \mathbf{j} - \mathbf{k}.$$

Calculate

- (a) $\mathbf{b} \times \mathbf{c}$, (3)
- (b) $\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c})$, (2)
- (c) the area of triangle OBC , (2)
- (d) the volume of the tetrahedron $OABC$. (1)

$$(a) \quad \underline{\mathbf{b}} \times \underline{\mathbf{c}} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 3 & -1 & 1 \\ 2 & 1 & -1 \end{vmatrix} = \begin{vmatrix} 3 & -1 & 1 \\ 2 & 1 & -1 \end{vmatrix}$$

$$= \begin{pmatrix} 0 \\ 5 \\ 5 \end{pmatrix}$$

$$(b) \quad \underline{\mathbf{a}} \cdot (\underline{\mathbf{b}} \times \underline{\mathbf{c}}) = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} \cdot \begin{pmatrix} 0 \\ 5 \\ 5 \end{pmatrix} = 5$$



Question 2 continued

$$(c) \text{ Area } \triangle OBC = \frac{1}{2} |\underline{b} \times \underline{c}| = \frac{1}{2} \sqrt{0^2 + 5^2 + 5^2}$$

$$= \frac{5}{2} \sqrt{2} \text{ units}^2$$

$$(d) \text{ Vol} = \frac{1}{6} |\underline{a} \cdot (\underline{b} \times \underline{c})| = \frac{1}{6} \times 5$$

$$= \frac{5}{6} \text{ units}^3$$

3.

$$M = \begin{pmatrix} 6 & 1 & -1 \\ 0 & 7 & 0 \\ 3 & -1 & 2 \end{pmatrix}$$

(a) Show that 7 is an eigenvalue of the matrix M and find the other two eigenvalues of M . (5)

(b) Find an eigenvector corresponding to the eigenvalue 7. (4)

$$(a). \quad M - \lambda I = \begin{pmatrix} 6-\lambda & 1 & -1 \\ 0 & 7-\lambda & 0 \\ 3 & -1 & 2-\lambda \end{pmatrix}$$

$$\det(M - \lambda I) = 0$$

$$\Rightarrow \det(M - \lambda I) = (6-\lambda)(7-\lambda)(2-\lambda) - 1(0)$$

$$+ -1(0 - 3(7-\lambda))$$

$$= (6-\lambda)(7-\lambda)(2-\lambda) - (-21 + 3\lambda)$$

$$= (6-\lambda)(7-\lambda)(2-\lambda) + 3(7-\lambda)$$

$$= (7-\lambda)((6-\lambda)(2-\lambda) + 3)$$

$$= (7-\lambda)(\lambda^2 - 8\lambda + 15) = 0$$

$$\Rightarrow 7-\lambda=0 \quad \therefore \lambda=7 \text{ is an eigenvalue}$$

$$\text{Also } \lambda^2 - 8\lambda + 15 = 0$$

$$\Rightarrow (\lambda-5)(\lambda-3) = 0 \quad \therefore \lambda=5, \lambda=3 \text{ are eigenvalues}$$



Question 3 continued

$$(b) \quad Ax = \lambda x$$

$$\Rightarrow \begin{pmatrix} 6 & 1 & -1 \\ 0 & 7 & 0 \\ 3 & -1 & 2 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} \lambda x \\ \lambda y \\ \lambda z \end{pmatrix}$$

$$\therefore \begin{pmatrix} 6x + y - z \\ 7y \\ 3x - y + 2z \end{pmatrix} = \begin{pmatrix} \lambda x \\ \lambda y \\ \lambda z \end{pmatrix}$$

$$\therefore 6x + y - z = \lambda x \Rightarrow x = y - z$$

$$7y = \lambda y$$

$$3x - y + 2z = \lambda z \Rightarrow 3x - y = 5z$$

$$\Rightarrow z = \frac{3}{5}x - \frac{y}{5}$$

$$\therefore x = y - \frac{3}{5}x + \frac{y}{5} \Rightarrow \frac{8}{5}x = \frac{6}{5}y$$

$$\Rightarrow x = \frac{3}{4}y$$

$$z = y - x = \frac{4}{3}x - x = \frac{1}{3}x \Rightarrow z = \frac{1}{3}x$$

$$x=1 \Rightarrow y = \frac{4}{3} \quad z = \frac{1}{3}$$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \frac{1}{3} \begin{pmatrix} 3 \\ 4 \\ 1 \end{pmatrix}, \quad \text{Let } \alpha = \begin{pmatrix} 3 \\ 4 \\ 1 \end{pmatrix} \text{ be the eigen vector}$$

4. Given that $y = \operatorname{arsinh}(\sqrt{x})$, $x > 0$,

(a) find $\frac{dy}{dx}$, giving your answer as a simplified fraction. (3)

(b) Hence, or otherwise, find

$$\int_{\frac{1}{4}}^4 \frac{1}{\sqrt{x(x+1)}} dx,$$

giving your answer in the form $\ln\left(\frac{a+b\sqrt{5}}{2}\right)$, where a and b are integers. (6)

4 (a). $y = \operatorname{arsinh}(\sqrt{x})$

$$\therefore \sinh y = \sqrt{x}$$

$$\begin{aligned} c^2 - s^2 &= 1 \\ c^2 &= 1 + s^2 \end{aligned}$$

$$\therefore \frac{dy}{dx} \cosh y = \frac{1}{2} x^{-\frac{1}{2}}$$

$$\begin{aligned} c^2 + s^2 &= 2 \\ c^2 - 1 + s^2 &= 2 \end{aligned}$$

$$\therefore \frac{dy}{dx} = \frac{1}{2\sqrt{x} \cosh y}$$

$$\therefore \frac{dy}{dx} = \frac{1}{2\sqrt{x} \sqrt{1 + \sinh^2 y}} = \frac{1}{2\sqrt{x} \sqrt{1 + x}}$$

$$\therefore \frac{dy}{dx} = \frac{1}{2\sqrt{x+x^2}}$$

(b) $\int_{\frac{1}{4}}^4 \frac{1}{\sqrt{x(x+1)}} dx = 2 \int_{\frac{1}{4}}^4 \frac{1}{2\sqrt{x+x^2}} dx$

$$= 2 \int_{\frac{1}{4}}^4 \frac{d}{dx} (\operatorname{arsinh} \sqrt{x}) dx$$



$$= 2 [\operatorname{arsinh} \sqrt{x}]_{\frac{1}{4}}^4$$

$$= 2 (\operatorname{arsinh} 2 - \operatorname{arsinh} \frac{1}{2})$$

$$= 2 \left(\ln(\sqrt{5} + 2) - \ln\left(\frac{1}{2} + \sqrt{\frac{5}{4}}\right) \right)$$

$$= 2 \left(\ln(2 + \sqrt{5}) - \ln\left(\frac{1 + \sqrt{5}}{2}\right) \right)$$

$$= 2 \ln\left(\frac{3 + \sqrt{5}}{2}\right) = \ln \left[\left(\frac{3 + \sqrt{5}}{2}\right)^2 \right]$$

$$= \ln\left(\frac{7 + 3\sqrt{5}}{2}\right)$$

5.

$$I_n = \int_0^5 \frac{x^n}{\sqrt{25-x^2}} dx, \quad n \geq 0$$

(a) Find an expression for $\int \frac{x}{\sqrt{25-x^2}} dx$, $0 \leq x \leq 5$.

(2)

(b) Using your answer to part (a), or otherwise, show that

$$I_n = \frac{25(n-1)}{n} I_{n-2} \quad n \geq 2$$

(5)

(c) Find I_4 in the form $k\pi$, where k is a fraction.

(4)

$$5(a). \int \frac{x}{\sqrt{25-x^2}} dx = \int x (25-x^2)^{-1/2}$$

$$= -\frac{1}{2} \int -2x (25-x^2)^{-1/2} dx$$

$$= -\frac{1}{2} \left(\frac{(25-x^2)^{1/2}}{1/2} \right) + C$$

$$= -\sqrt{25-x^2} + C$$

$$(b) I_n = \int_0^5 \frac{x^n}{\sqrt{25-x^2}} dx = \int_0^5 \frac{x^{n-1} \cdot x}{\sqrt{25-x^2}} dx$$

$$\text{Let } u = x^{n-1} \quad u' = (n-1)x^{n-2}$$

$$v' = \frac{x}{\sqrt{25-x^2}}$$

$$v = -\sqrt{25-x^2}$$



Question 5 continued

$$\therefore I_n = \left[-x^{n-1} \sqrt{25-x^2} \right]_0^5 + (n-1) \int_0^5 x^{n-2} \sqrt{25-x^2} \, dx$$

$$\therefore I_n = (n-1) \int_0^5 x^{n-2} (25-x^2)^{-\frac{1}{2}} (25-x^2) \, dx$$

$$\therefore I_n = (n-1) \int_0^5 25x^{n-2} (25-x^2)^{-\frac{1}{2}} - x^n (25-x^2)^{-\frac{1}{2}} \, dx$$

$$\therefore I_n = (n-1) (25 I_{n-2} - I_n)$$

$$\therefore I_n = 25(n-1) I_{n-2} - (n-1) I_n$$

$$\therefore n I_n = 25(n-1) I_{n-2}$$

$$\Rightarrow I_n = \frac{25(n-1)}{n} I_{n-2} \text{ as required.}$$

$$(C) I_4 = \frac{75}{4} I_2 = \frac{75}{4} \left(\frac{25}{2} I_0 \right)$$

$$\therefore I_4 = \frac{1875}{8} \int_0^5 \frac{1}{\sqrt{25-x^2}} \, dx = \frac{1875}{8} \left[\operatorname{arcsinh} \frac{x}{5} \right]_0^5$$

$$= \frac{1875}{8} (\operatorname{arcsinh} 1)$$

$$= \frac{1875}{16} \pi$$

6. The hyperbola H has equation $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$, where a and b are constants.

The line L has equation $y = mx + c$, where m and c are constants.

(a) Given that L and H meet, show that the x -coordinates of the points of intersection are the roots of the equation

$$(a^2m^2 - b^2)x^2 + 2a^2mcx + a^2(c^2 + b^2) = 0 \quad (2)$$

Hence, given that L is a tangent to H ,

(b) show that $a^2m^2 = b^2 + c^2$. (2)

The hyperbola H' has equation $\frac{x^2}{25} - \frac{y^2}{16} = 1$.

(c) Find the equations of the tangents to H' which pass through the point $(1, 4)$. (7)

6 (a). $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ & $y = mx + c$ @ intersection.

$$\frac{x^2}{a^2} - \frac{(mx+c)^2}{b^2} = 1$$

$$\frac{x^2}{a^2} - \frac{m^2x^2 + 2mcx + c^2}{b^2} = 1$$

$$\frac{x^2b^2}{a^2b^2} - \frac{a^2(m^2x^2 + 2amcx + a^2c^2)}{a^2b^2} = 1$$

$$x^2b^2 - a^2m^2x^2 - 2a^2mcx - a^2c^2 = a^2b^2$$

$$\Rightarrow a^2m^2x^2 - x^2b^2 + 2a^2mcx + a^2c^2 + a^2b^2 = 0$$

$$\therefore (a^2m^2 - b^2)x^2 + 2a^2mcx + a^2(c^2 + b^2) = 0$$

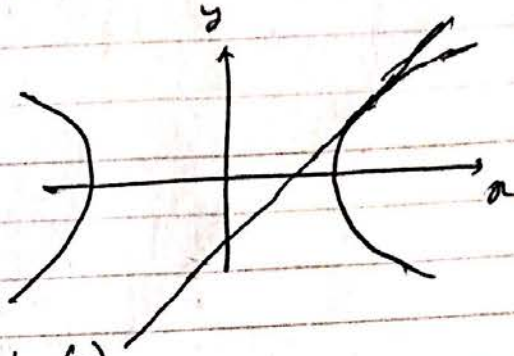
as required.



Question 6 continued

(b)

L and H intersect once.



\therefore Quadratic eqn in part (a) has only one solution. \Rightarrow discriminant = 0

$$\therefore (2a^2mc)^2 - 4(a^2m^2 - b^2)(a^2(c^2 + b^2)) = 0$$

$$\therefore 4a^4m^2c^2 - 4a^2(a^2m^2 - b^2)(c^2 + b^2) = 0$$

$$\therefore 4a^4m^2c^2 - 4a^2(a^2m^2c^2 + a^2m^2b^2 - b^2c^2 - b^4) = 0$$

$$\therefore 4a^4m^2c^2 - 4a^4m^2c^2 - 4a^4m^2b^2 + 4a^2b^2c^2 + 4a^2b^4 = 0$$

$$\therefore -4a^4m^2b^2 + 4a^2b^2c^2 + 4a^2b^4 = 0$$

$$\therefore 4a^2b^2(c^2 - a^2m^2 + b^2) = 0$$

$$\therefore c^2 - a^2m^2 + b^2 = 0 \Rightarrow c^2 + b^2 = a^2m^2$$

as required.

(c) $\frac{x^2}{25} - \frac{y^2}{16} = 1 \Rightarrow a^2 = 25, b^2 = 16$
 $a^2m^2 = b^2 + c^2$

$$\therefore 25m^2 = 16 + c^2$$

$$4a^2m^2 - b^2 = 25m^2 - 16$$

$$(1, 4) \Rightarrow 4 = m + c$$

$$\Rightarrow c = 4 - m$$

$$\therefore 25m^2 = 16 + (4 - m)^2$$

$$\therefore 25m^2 = m^2 - 8m + 32$$

$$\therefore \cancel{m^2 - 8m}$$

$$24m^2 + 8m - 32 = 0$$

$$\therefore \cancel{24} 3m^2 + m - 4 = 0$$

$$\cancel{(3m - 4)(m + 1)}$$

$$\cancel{m} \cdot (m - 1)(3m + 4) = 0$$

$$\therefore m = 1 \quad \& \quad m = -\frac{4}{3}$$

$$\therefore c = 4 - 1 = 3$$

$$\& \quad c = 4 + \frac{4}{3} = \frac{16}{3}$$

$$\therefore y = x + 3$$

$$\& \quad y = -\frac{4}{3}x + \frac{16}{3}$$

7. The lines l_1 and l_2 have equations

$$\mathbf{r} = \begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix} + \lambda \begin{pmatrix} -1 \\ 3 \\ 4 \end{pmatrix} \text{ and } \mathbf{r} = \begin{pmatrix} \alpha \\ -4 \\ 0 \end{pmatrix} + \mu \begin{pmatrix} 0 \\ 3 \\ 2 \end{pmatrix}.$$

If the lines l_1 and l_2 intersect, find

(a) the value of α , (4)

(b) an equation for the plane containing the lines l_1 and l_2 , giving your answer in the form $ax + by + cz + d = 0$, where a, b, c and d are constants. (4)

For other values of α , the lines l_1 and l_2 do not intersect and are skew lines.

Given that $\alpha = 2$,

(c) find the shortest distance between the lines l_1 and l_2 . (3)

7. (a). $\underline{r} = \begin{pmatrix} 1-\lambda \\ 3\lambda-1 \\ 2+4\lambda \end{pmatrix}$ & $\underline{r} = \begin{pmatrix} \alpha \\ -4+3\mu \\ 2\mu \end{pmatrix}$

$$3\lambda - 1 = -4 + 3\mu$$

$$\& \quad 2 + 4\lambda = 2\mu \Rightarrow 3 + 6\lambda = 3\mu$$

$$\therefore 3\lambda - 1 = -4 + 3 + 6\lambda$$

$$\Rightarrow 3\lambda = 0$$

$$\Rightarrow \lambda = 0 \quad \mu = 1$$

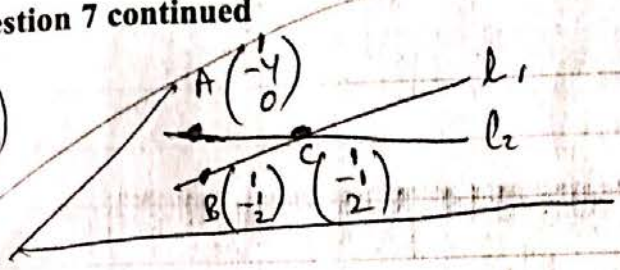
$$1 - \lambda = \alpha \Rightarrow \alpha = 1$$

(b)



Question 7 continued

(b)



Let E be intersection
Intersection @ C

Let A be point on l_1 : $\begin{pmatrix} -4 \\ 0 \end{pmatrix}$
 Let B be point on l_2 : $\begin{pmatrix} -1/2 \end{pmatrix}$

$$\vec{AC} = \begin{pmatrix} -1/2 \\ 2 \end{pmatrix} - \begin{pmatrix} -4 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 3 \\ 2 \end{pmatrix}$$

$$\vec{BC} = \begin{pmatrix} -1/2 \\ 2 \end{pmatrix} - \begin{pmatrix} -1/2 \end{pmatrix}$$

$$(b). \hat{n} = \begin{pmatrix} -1 \\ 3 \\ 4 \end{pmatrix} \times \begin{pmatrix} 0 \\ 3 \\ 2 \end{pmatrix} = \begin{pmatrix} -6 \\ 2 \\ -3 \end{pmatrix}$$

$$\therefore \hat{n} \cdot \vec{r} = \hat{n} \cdot \begin{pmatrix} -6 \\ 2 \\ -3 \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix} \cdot \begin{pmatrix} -6 \\ 2 \\ 3 \end{pmatrix}$$

$$\therefore -6x + 2y - 3z = -6 - 2 - 6$$

$$\therefore -6x + 2y - 3z = -14$$

$$\therefore 6x - 2y + 3z - 14 = 0$$

Question 7 continued

Use:

$$d = \frac{|(a-c) \cdot (b \times d)|}{|b \times d|}$$

(c) ~~f~~ -

$$d = \frac{\left| \left[\begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix} - \begin{pmatrix} 2 \\ -4 \\ 0 \end{pmatrix} \right] \cdot \begin{bmatrix} -6 \\ 2 \\ -3 \end{bmatrix} \right|}{\sqrt{6^2 + 2^2 + 3^2}}$$

$$\therefore d = \frac{\begin{pmatrix} -1 \\ 3 \\ 2 \end{pmatrix} \cdot \begin{pmatrix} -6 \\ 2 \\ -3 \end{pmatrix}}{\sqrt{6^2 + 2^2 + 3^2}} = \frac{6}{7}$$

$$\therefore \text{distance} = \frac{6}{7}$$

8. A curve, which is part of an ellipse, has parametric equations

$$x = 3 \cos \theta, \quad y = 5 \sin \theta, \quad 0 \leq \theta \leq \frac{\pi}{2}.$$

The curve is rotated through 2π radians about the x -axis.

(a) Show that the area of the surface generated is given by the integral

$$k\pi \int_0^\alpha \sqrt{(16c^2 + 9)} dc, \quad \text{where } c = \cos \theta,$$

and where k and α are constants to be found.

(6)

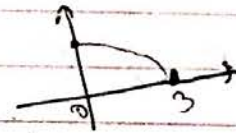
(b) Using the substitution $c = \frac{3}{4} \sinh u$, or otherwise, evaluate the integral, showing all of your working and giving the final answer to 3 significant figures.

(5)

$$(a) \quad x = 3 \cos \theta \quad \therefore \frac{\partial x}{\partial \theta} = -3 \sin \theta \quad \therefore \left(\frac{\partial x}{\partial \theta}\right)^2 = 9 \sin^2 \theta$$

$$y = 5 \sin \theta \quad \therefore \frac{\partial y}{\partial \theta} = 5 \cos \theta \quad \therefore \left(\frac{\partial y}{\partial \theta}\right)^2 = 25 \cos^2 \theta$$

$$S = 2\pi \int_{\pi/2}^0 y \sqrt{\left(\frac{\partial x}{\partial \theta}\right)^2 + \left(\frac{\partial y}{\partial \theta}\right)^2} d\theta$$



Let $s = \sin \theta$ Let $c = \cos \theta$

$$\therefore S = 2\pi \int_{\pi/2}^0 5s \sqrt{9c^2 + 25c^2} d\theta$$

$$= 10\pi \int_{\pi/2}^0 s \sqrt{9 - 9c^2 + 25c^2} d\theta$$

$$S = -10\pi \int_{\pi/2}^0 s \sqrt{9 + 16c^2} d\theta$$

Limits from $\frac{\pi}{2}$ to 0
gives -ve s
 \therefore add a '-' sign
in front of integral

Question 5 continued

Now let c represent a variable:

~~$S = 10\pi$~~ $C = \cos\theta$

$$\frac{dc}{d\theta} = -\sin\theta$$

$$\therefore \frac{d\theta}{dc} = -\frac{1}{\sin\theta} dc = -\frac{1}{s} dc$$

~~$S = 10\pi \int_{\pi}^{\pi/2}$~~ Consider limits:
 ~~$c = 0 \Rightarrow \cos\theta = 0$~~

$$\begin{aligned} C = \cos(0) &= 1 \\ C = \cos(\pi/2) &= 0 \end{aligned} \Rightarrow S = \int_0^1$$

$$\therefore S = -10\pi \int_{\pi/2}^0 \sqrt{9+16c^2} d\theta = -10\pi \int_0^1 \sqrt{9+16c^2} \cdot \frac{-1}{s} dc$$

$$\therefore S = -10\pi \int_0^1 -\sqrt{16c^2+9} dc$$

$$\therefore S = +10\pi \int_0^1 \sqrt{16c^2+9} dc \quad a=1$$

(Total 10 marks)

Q5

$$(b) \quad 10\pi \int_0^1 \sqrt{16 \left(\frac{3}{4} \sinh u\right)^2 + 9} \, dc$$

$$c = \frac{3}{4} \sinh u$$

$$\therefore \frac{dc}{du} = \frac{3}{4} \cosh u$$

$$\therefore dc = \frac{3}{4} \cosh u \, du$$

$$\therefore S = 10\pi \int \sqrt{9 \sinh^2 u + 9} \cdot \frac{3}{4} \cosh u \, du$$

$$= 10\pi \int \frac{9}{4} \sqrt{\sinh^2 u + 1} \cosh u \, du$$

$$= 10\pi \int \frac{9}{4} \cosh^2 u \, du$$

$$= \frac{90}{8} \pi \int \cosh 2u + 1 \, du$$

$$= \frac{90}{8} \pi \left[\frac{1}{2} \sinh 2u + u \right]$$

$$\frac{c^2 - 5^2}{c^4 + 5^2} = \cosh 2x$$

$$\frac{c^2 + c^2 - 1}{2c^2 - 1}$$

Consider limits: $c = \frac{3}{4} \sinh(1) = \frac{3}{4} \frac{1}{2}(e^1 - e^{-1})$
 $= \frac{3}{8}(e - \frac{1}{e})$

~~$c = \frac{3}{4} \sinh$~~ $1 = \frac{3}{4} \sinh u \Rightarrow u = \operatorname{arsinh} \frac{4}{3} = \ln 3$
 $0 = \frac{3}{4} \sinh u \Rightarrow u = \operatorname{arsinh}(0) = 0$

$$\dots \frac{90}{8} \pi \int \left[\frac{1}{2} \sinh 2u + u \right]_{\operatorname{arsinh}(0) = \ln(1) = 0}^{\operatorname{arsinh} \frac{4}{3} = \ln \left(\frac{4}{3} + \sqrt{\frac{4}{9} + 1} \right) = \ln 3}$$

$$= \frac{90}{8} \pi \left[\frac{1}{2} \sinh 2u + u \right]_0^{\ln 3}$$

$$= \frac{90}{8} \pi \left(\frac{1}{2} \sinh \ln 9 + \ln 3 \right)$$

$$= \frac{90}{8} \pi \left[\frac{1}{2} \frac{1}{2} (e^{\ln 9} - e^{-\ln 9}) + \ln 3 \right]$$

$$= \frac{90}{8} \pi \left[\frac{1}{4} \left(9 - \frac{1}{9} \right) + \ln 3 \right]$$

$$= 117.36 \dots = \underline{\underline{117}} \quad \underline{\underline{(3sf)}}$$